J. Math. Anal. Appl. 373 (2011) 248–251

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A uniqueness result for a nonlocal equation of Kirchhoff type and some related open problem

Giovanni Anello

Department of Mathematics, University of Messina, 98166 S. Agata, Messina, Italy

ARTICLE INFO

Article history:
Received 17 March 2010
Available online 15 July 2010
Submitted by Goong Chen

Keywords:
Elliptic boundary value problems
Nonlocal equations
Uniqueness
Variational methods

ABSTRACT

We study the uniqueness of solution for the following boundary value problem involving a nonlocal equation of Kirchhoff type

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda f(u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

Here, Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary, a, b, λ are positive real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In particular, we give an answer to an open problem recently proposed by B. Ricceri.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let λ, μ be positive real parameters. Very recently, in [4] B. Ricceri, by means of an abstract variational result by the same author, has established a multiplicity theorem for the following Kirchhoff-type problem

$$\begin{cases} -K\left(\displaystyle\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0 \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $K : [0, \infty[\to \mathbb{R}$ is a continuous function, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions satisfying the subcritical growth condition

$$\alpha)\quad \begin{array}{ll} \displaystyle\sup_{t\in\mathbb{R}} \frac{|f(\cdot,t)| + |g(\cdot,t)|}{1+|t|^q} \in L^\infty(\Omega) & \text{for some } q \in\,]0, +\infty[\text{ if } n \geqslant 2 \\ & \text{with } q < \dfrac{N+2}{N-2} \text{ if } n > 2; \\ \displaystyle\sup_{t\in[-r,r]} \left(\left|f(\cdot,t)\right| + \left|g(\cdot,t)\right|\right) \in L^1(\Omega) & \text{for all } r > 0 \text{ if } n = 1. \end{array}$$

E-mail address: ganello@unime.it.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.07.019

For our convenience, we report here the main result of [4].

Theorem A. *(See Theorem 1 of [4].) Let* $K:[0,\infty[\,\to\mathbb{R}$ *be a continuous function and let* $f:\Omega\times\mathbb{R}\to\mathbb{R}$ *be a Carathéodory function satisfying condition* α*). Put* $\tilde{K}(t)=\int_0^t K(s)\,ds$ *for* $t\geqslant 0$ *and* $F(x,t)=\int_0^t f(x,s)\,ds$ *for* $(x,t)\in\Omega\times\mathbb{R}$*. Assume the following conditions:*

a_1) $\sup_{u\in W_0^{1,2}(\Omega)}\int_\Omega F(u,u(x))\,dx>0$;
a_2) $\inf_{t\geqslant 0}K(t)>0$;
a_3) *for some* $\alpha>0$ *one has* $\liminf_{t\to+\infty}\frac{\tilde{K}(t)}{t^\alpha}>0$;
a_4) *there exists a continuous function* $h:[0,+\infty[\,\to\mathbb{R}$ *such that* $h(tK(t^2))=t$ *for all* $t\geqslant 0$;
a_5) $\limsup_{t\to 0}F(x,t)t^{-2}\leqslant 0$ *a.e. in* Ω;
a_6) $\limsup_{|t|\to+\infty}F(x,t)t^{-2\alpha}\leqslant 0$ *a.e. in* Ω.

Then, for every

$$c>\inf\left\{\frac{\tilde{K}(\int_\Omega|\nabla u|^2\,dx)}{2F(x,u(x))\,dx}:\ u\in W_0^{1,2}(\Omega),\ \int_\Omega F\big(x,u(x)\big)\,dx>0\right\}\quad\text{and}\quad d>c$$

there exists a number $r>0$ *with the following property: for every* $\lambda\in[c,d]$ *and every Carathéodory function* $g:\Omega\times\mathbb{R}\to\mathbb{R}$ *satisfying condition* α*), there exists* $\delta>0$ *such that, for each* $\mu\in\,]0,\delta[$*, problem* $(P_{\lambda,\mu})$ *admits at least three weak solutions whose norms in* $W_0^{1,2}(\Omega)$ *are less than* r*.*

Recently, some papers has been devoted to the study of boundary value problems of Kirchhoff type in the particular case of $K(t)=a+bt$ for all $t\geqslant 0$, where a, b are fixed positive real numbers (see references of [4] for instance). The emphasis given to this case is due to the fact that with the previous choice of K an equation of the type

$$-K\left(\int_\Omega\left|\nabla u(x)\right|^2dx\right)\Delta u=\varphi(x,u)$$

is, for the stationary case, exactly the one originally proposed by Kirchhoff in [3]. This is an extension of the D'Alembert's wave equation for free vibrations of elastic strings which takes into account the length changes of the string produced by transverse vibrations. For this reason, it becomes interesting to give the particular statement that Theorem A takes with the above choice of K. For simplicity, we consider the function f independent of the first variable. As showed in [4], conditions a_2) and a_4) are automatically satisfied as well as condition a_3) by choosing $\alpha\leqslant 2$. Moreover, since f fulfills condition α), it is easy to check that condition a_6) holds with $\alpha\geqslant\frac{n}{n-2}$ if $n\geqslant 3$. In particular, choosing $\alpha=2$ and $n\geqslant 4$ we have that conditions a_2), a_4) and a_6) hold. Thus, in this case, to get all the assumptions of Theorem 1 satisfied it is enough to impose on f (besides α)) only the conditions a_1) and a_5) where this latter, for f independent of x, takes the simpler statement $\sup_{t\in\mathbb{R}}\int_0^t f(s)\,ds>0$.

Summarizing, we have the following result:

Theorem B. *(See Theorem 2 of [4].) Let* $n\geqslant 4$*, let* $q\in\,]0,\frac{n+2}{n-2}[$ *and let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function such that*

$$\limsup_{|t|\to+\infty}\frac{|f(t)|}{|t|^q}<+\infty,\qquad\limsup_{t\to 0}\frac{\int_0^t f(s)\,ds}{t^2}\leqslant 0\quad\text{and}\quad\sup_{t\in\mathbb{R}}\int_0^t f(s)\,ds>0.$$

Then, if $K(t)=a+bt$ *for* $t\geqslant 0$ *and with* $a,b>0$*, conclusion of Theorem* 1 *holds.*

The statement of Theorem B leads to the following open problem proposed by the author of [4].

Problem. Does Theorem 2 hold for $n=3$?

The aim of this short note is to give a negative answer to the above problem (which comes from Theorem 1 below) and, at the same time, to propose some further open problem.

2. The result

Let Ω be the open ball in $\mathbb{R}^3$ centered at 0 with radius $R>0$ and let $q\in\,]3,5[$. We consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(t)=t^q$ if $t\geqslant 0$, $f(t)=0$ if $t<0$. We have the following result:

Theorem 1. *Let a, b, λ be three positive real numbers. The problem*

$$\begin{cases} -\left(a+b\int\limits_{\Omega}|\nabla u|^2\,dx\right)\Delta u=\lambda f(u) & \textit{in } \Omega,\\ u_{|\partial\Omega}=0 \end{cases} \tag{P}$$

admits a unique nonzero weak solution.

Proof. First of all, we note that by the Strong Maximum Principle every nonzero solution of problem (P) must be positive. For every $u \in W_0^{1,2}(\Omega)$, we put

$$\|u\|=\left(\int\limits_{\Omega}|\nabla u|^2\,dx\right)^{\frac{1}{2}}.$$

The weak solutions of problem (P) are exactly the critical points of the functional

$$I(u)=\frac{1}{2}\left(a\|u\|^2+\frac{b}{2}\|u\|^4\right)-\lambda\int\limits_{\Omega}\left(\int\limits_{0}^{u(x)}f(t)\,dt\right)dx,\quad u\in W_0^{1,2}(\Omega).$$

Denoting by C the best embedding constant of $W_0^{1,2}(\Omega)$ in $L^{q+1}(\Omega)$, for every $r>0$ we have

$$\inf_{\|u\|=r}I(u)=\frac{1}{2}\left(ar^2+\frac{b}{2}r^4\right)-\lambda C^q r^{q+1}.$$

Hence, being $q+1>4$,

$$\liminf_{\|u\|\to+\infty}I(u)=-\infty$$

and there exists $r_0>0$ such that

$$\inf_{\|u\|=r_0}I(u)>0.$$

Thus, I has a Mountain Pass geometry. Moreover, by standard arguments, we have that I satisfies the Palais–Smale condition. Therefore, I has a nonzero critical point $u_1 \in W_0^{1,2}(\Omega)$. Let us to show that u_1 is unique. Assume that $u_2 \in W_0^{1,2}(\Omega)$ is another nonzero critical point. Note that u_i $(i=1,2)$ is a weak solution of the problem

$$\begin{cases} -\Delta u=\dfrac{\lambda}{a+b\|u_i\|^2}f(u) & \text{in } \Omega,\\ u_{|\partial\Omega}=0. \end{cases}$$

Now, in [1] it is proved that the problem

$$\begin{cases} -\Delta u=f(u) & \text{in } \Omega,\\ u_{|\partial\Omega}=0 \end{cases}$$

has a unique nonzero (and positive) solution $v_1 \in W_0^{1,2}(\Omega)$. This implies that for all $\lambda>0$, the function $v_\lambda=\lambda^{\frac{1}{1-q}}v_1$ is the unique nonzero solution of the problem

$$\begin{cases} -\Delta u=\lambda f(u) & \text{in } \Omega,\\ u_{|\partial\Omega}=0. \end{cases} \tag{P_λ}$$

In particular, for $\lambda_1,\lambda_2>0$, v_{λ_1} and v_{λ_2} are related by

$$v_{\lambda_1}=\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-q}}v_{\lambda_2}.$$

Therefore, since u_i is the nonzero solutions of problem (P_{λ_i}) with

$$\lambda_i=\frac{\lambda}{a+b\int_\Omega|\nabla u_i|^2\,dx},$$

we have

$$u_1 = \left(\frac{a + b\|u_2\|^2}{a + b\|u_1\|^2} \right)^{\frac{1}{1-q}} u_2 \tag{1}$$

from which we get

$$\|u_1\| = \left(\frac{a + b\|u_2\|^2}{a + b\|u_1\|^2} \right)^{\frac{1}{1-q}} \|u_2\|$$

equivalent to

$$a\|u_1\|^{1-q} + b\|u_1\|^{3-q} = a\|u_2\|^{1-q} + b\|u_2\|^{3-q}. \tag{2}$$

Since $q > 3$, the function $t > 0 \to at^{1-q} + bt^{3-q}$ is strictly decreasing. Consequently, from (2) it follows $\|u_1\| = \|u_2\|$ and so, from (1), $u_1 = u_2$. $\square$

The proof of Theorem 1 is based on the uniqueness result of [1] whose validity depends on the shape of the set Ω. Indeed, if Ω is ring-shaped the previous uniqueness results does not hold any longer at least for $q + 1$ near the critical exponent as showed in [2]. So, knowing whether problem (P) may have more than one solution for particular domains Ω remains an open problem.

References

[1] R. Adimurthi, S.L. Yadava, An elementary proof of the uniqueness of positive radial solutions of a quasilinear Dirichlet problem, Arch. Ration. Mech. Anal. 127 (1994) 219–229.

[2] J. Garcia, I. Peral, On limits of solutions of elliptic problems with nearly critical exponent, Comm. Partial Differential Equations 17 (11–12) (1992) 2113–2126.

[3] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[4] B. Ricceri, On an elliptic Kirchhoff-type problem depending on two parameters, J. Global Optim. 46 (2010) 543–549.